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Rate of convergence of Chlodowsky type Durrmeyer Jakimovski-Leviatan operators

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Abstract

In this paper, we determine the rate of pointwise convergence of the Chlodowsky type Durrmeyer Jakimovski-Leviatan operators $L_n^*(f,x)$ for functions of bounded variation. We use some methods and techniques of probability theory to prove our main result.

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1 Introduction and preliminaries

Very recently, some authors studied some linear positive operators and obtained the rate of convergence for functions of bounded variation. For example, Bojanic and Vuilleumier [1] estimated the rate of convergence of Fourier Legendre series of functions of bounded variation on the interval $[0,1]$, Cheng F. [3] estimated the rate of convergence of Bernstein polynomials of functions of bounded variation on the interval $[0,1]$, Zeng and Chen [16] estimated the rate of convergence of Durrmeyer type operators for functions of bounded variation on the interval $[0,1]$. On the other hand, in recent years, there is an increasing interest in modifying linear operators so that the new versions reproduce some basic functions, e.g. summation-integral type operators [11] and [12], Bézier variant of the Bleimann-Butzer-Hahn operators [13], Szász-Bézier integral operators [15]; and some q-analogous [8], [9], [10], [14]. This line of work originated with a paper by Jakimovski and Leviatan [5], who introduced a Favard-Szász type operators P_n, by using Appell polynomials $p_k(x)$, $k\geq 0$, defined by

$$g(u)e^{ux}=\sum_{k=0}^{\infty}p_k(x)u^k, \tag{1.1}$$

where $g(u)=\sum_{k=0}^{\infty}a_ku^k$ is an analytic function in the disk $\mid u\mid< R$, $R>1$ and $g(1)\neq 0$,

$$P_n(f;x)=\frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty}p_k(nx)f(\frac{k}{n}), \tag{1.2}$$

if $g(u)=1$, by (1.1) we obtain $p_k(x)=\frac{(x)^k}{k!}$ and we obtain Szász-Mirakjan operators:

$$S_n(f;x)=e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f(\frac{k}{n}).$$

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Chlodowsky type generalization of Jakimovski-Leviatan operators is investigated in [2]. These operators are defined as

$$P_n^*(f;x) = \frac{e^{\frac{-nx}{b_n}}}{g(1)} \sum_{k=0}^{\infty} p_k(\frac{nx}{b_n}) f(\frac{k}{n} b_n), \tag{1.3}$$

with b_n a positive increasing sequence with the properties

$$\lim_{n\to\infty} b_n = \infty, \qquad \lim_{n\to\infty} \frac{b_n}{n} = 0.$$

Works on Chlodowsky operators are fewer, since they are defined on an unbounded interval $[0,\infty)$. Durrmeyer type Jakimovski-Leviatan operators L_n is introduced by Ali Karaisa in [6]. The polynomial $L_n(f;x)$ is defined by

$$L_n(f;x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} \frac{p_k(nx)}{B(n+1,k)} \int_0^{\infty} \frac{t^{k-1}}{(1+t)^{n+k+1}} f(t) dt, \qquad x \geq 0, \tag{1.4}$$

where $B(n+1,k)$ is beta function defined by

$$B(x,y) = \int_0^{\infty} \frac{t^{x-1}}{(1+t)^{x+y}} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \qquad x,y > 0.$$

2 Construction of operators and auxiliary results

For all real valued continuous and bounded functions f on $[0,\infty)$, we define the Chlodowsky [7] type Durrmeyer Jakimovski-Leviatan operators $L_n^* : BV[0,\infty) \to W$ as follows:

$$L_n^*(f;x) = \sum_{k=0}^{\infty} P_{n,k}(\frac{x}{b_n}) \int_0^{\infty} b_{n,k}(\frac{t}{b_n}) f(t) dt, \qquad x \in [0,\infty), \tag{2.1}$$

where $P_{n,k}(\frac{x}{b_n}) = \frac{e^{\frac{-nx}{b_n}}}{g(1)} p_k(\frac{nx}{b_n})$, $b_{n,k}(\frac{t}{b_n}) = \frac{1}{b_n} \frac{1}{B(n+1,k)} \frac{(\frac{t}{b_n})^{k-1}}{(1+\frac{t}{b_n})^{n+k+1}}$, $W = \{P : [0,\infty) \to R\}$ is a polynomial functions set.

In this section, we give certain results which are necessary to prove our main theorem.

Lemma 2.1. By (1.1), we obtain that

(i) $\sum_{k=0}^{\infty} p_k(\frac{nx}{b_n}) = e^{\frac{nx}{b_n}} g(1)$,

(ii) $\sum_{k=0}^{\infty} k p_k(\frac{nx}{b_n}) = e^{\frac{nx}{b_n}} [\frac{nx}{b_n} g(1) + g'(1)]$,

(iii) $\sum_{k=0}^{\infty} k^2 p_k(\frac{nx}{b_n}) = e^{\frac{nx}{b_n}} [\frac{n^2x^2}{b_n^2} g(1) + \frac{nx}{b_n}(2g'(1) + g(1)) + g''(1) + g'(1)]$.

Lemma 2.2. If $s \in N$ and $s \leq n$, then

$$L_n^*(t^s;x) = \frac{b_n^s e^{\frac{-nx}{b_n}}}{g(1)} \sum_{k=0}^{\infty} p_k(\frac{nx}{b_n}) \frac{(n-s)!(k+s-1)!}{n!(k-1)!} .$$

Proof.

$$\begin{aligned} L_n^*(t^s;x) &= \sum_{k=0}^{\infty} P_{n,k}(\frac{x}{b_n}) \int_0^{\infty} b_{n,k}(\frac{t}{b_n}) t^s dt \\ &= \frac{e^{\frac{-nx}{b_n}}}{b_n g(1)} \sum_{k=0}^{\infty} \frac{p_k(\frac{nx}{b_n})}{B(n+1,k)} \int_0^{\infty} \frac{(\frac{t}{b_n})^{k-1}}{(1+\frac{t}{b_n})^{n+k+1}} t^s dt. \end{aligned}$$

Set $u = \frac{t}{b_n}$,

$$\begin{aligned} L_n^*(t^s;x) &= \frac{b_n^s e^{\frac{-nx}{b_n}}}{g(1)} \sum_{k=0}^{\infty} \frac{p_k(\frac{nx}{b_n})}{B(n+1,k)} \int_0^{\infty} \frac{(u)^{k+s-1}}{(1+u)^{n+k+1}} dt \\ &= \frac{b_n^s e^{\frac{-nx}{b_n}}}{g(1)} \sum_{k=0}^{\infty} \frac{p_k(\frac{nx}{b_n})}{B(n+1,k)} B(k+s, n-s+1) \\ &= \frac{b_n^s e^{\frac{-nx}{b_n}}}{g(1)} \sum_{k=0}^{\infty} p_k(\frac{nx}{b_n}) \frac{(n-s)!(k+s-1)!}{n!(k-1)!}. \end{aligned}$$

Q.E.D.

Lemma 2.3. For $L_n^*(t^s;x)$, s=0, 1, 2, one have

(i) $L_n^*(1;x) = 1$,

(ii) $L_n^*(t;x) = x + \frac{b_n}{n} A$,

(iii) $L_n^*(t^2;x) = \frac{b_n^2}{n(n-1)} \{ \frac{n^2 x^2}{b_n^2} + \frac{nx}{b_n} B + C \}$,

where $A = \frac{g'(1)}{g(1)}, B = \frac{2g'(1)+2g(1)}{g(1)}, C = \frac{g''(1)+2g'(1)}{g(1)}$.

Proof. By Lemma (2.1)and Lemma (2.2), we get

(i)

$$\begin{aligned} L_n^*(t^s;x) &= \frac{b_n^s e^{\frac{-nx}{b_n}}}{g(1)} \sum_{k=0}^{\infty} p_k(\frac{nx}{b_n}) \frac{(n-s)!(k+s-1)!}{n!(k-1)!} \\ L_n^*(1;x) &= \frac{e^{\frac{-nx}{b_n}}}{g(1)} \sum_{k=0}^{\infty} p_k(\frac{nx}{b_n}) \\ L_n^*(1;x) &= 1 \ . \end{aligned}$$

(ii)

$$
\begin{aligned}
L_n^*(t^s;x) &= \frac{b_n^s e^{\frac{-nx}{b_n}}}{g(1)} \sum_{k=0}^{\infty} p_k(\frac{nx}{b_n}) \frac{(n-s)!(k+s-1)!}{n!(k-1)!} \\
L_n^*(t;x) &= \frac{b_n e^{\frac{-nx}{b_n}}}{g(1)} \sum_{k=0}^{\infty} p_k(\frac{nx}{b_n}) \frac{k}{n} \\
&= \frac{b_n}{n} \frac{e^{\frac{-nx}{b_n}}}{g(1)} \sum_{k=0}^{\infty} k p_k(\frac{nx}{b_n}) \\
&= x + \frac{b_n}{n} \frac{g'(1)}{g(1)} \\
&= x + \frac{b_n}{n} A.
\end{aligned}
$$

(iii)

$$
\begin{aligned}
L_n^*(t^s;x) &= \frac{b_n^s e^{\frac{-nx}{b_n}}}{g(1)} \sum_{k=0}^{\infty} p_k(\frac{nx}{b_n}) \frac{(n-s)!(k+s-1)!}{n!(k-1)!} \\
L_n^*(t^2;x) &= \frac{b_n^2 e^{\frac{-nx}{b_n}}}{g(1)} \sum_{k=0}^{\infty} p_k(\frac{nx}{b_n}) \frac{(k^2+k)}{n(n-1)} \\
&= \frac{b_n^2}{n(n-1)} \frac{e^{\frac{-nx}{b_n}}}{g(1)} \left\{ \sum_{k=0}^{\infty} k^2 p_k(\frac{nx}{b_n}) + \sum_{k=0}^{\infty} k p_k(\frac{nx}{b_n}) \right\} \\
&= \frac{b_n^2}{n(n-1)} \left\{ \frac{n^2x^2}{b_n^2} + \frac{nx}{b_n} \frac{(2g'(1)+2g(1))}{g(1)} + \frac{g''(1)+2g'(1)}{g(1)} \right\} \\
&= \frac{b_n^2}{n(n-1)} \left\{ \frac{n^2x^2}{b_n^2} + \frac{nx}{b_n} B + C \right\}.
\end{aligned}
$$

Q.E.D.

Lemma 2.4. The function $\mu_{n,m}(x)$, $m \in N^0$, can be defined as

$$\mu_{n,m}(x) = L_n^*((t-x)^m;x) = \sum_{k=0}^{\infty} P_{n,k}(\frac{x}{b_n}) \int_0^{\infty} b_{n,k}(\frac{t}{b_n})(t-x)^m dt.$$

Then,

(i) $\mu_{n,0}(x) = 1$,

(ii) $\mu_{n,1}(x) = \frac{b_n}{n} A$,

(iii) $\mu_{n,2}(x) = \frac{b_n^2}{n(n-1)} \{ \frac{n^2x^2}{b_n^2} + \frac{nx}{b_n} B + C \} - \frac{2b_n x}{n} A - x^2$.

Consequently, for each $x \in [0,\infty)$, $\mu_{n,m}(x) = O(n^{-(m+1)/2})$. For $x > 0$ by Lemma (2.4), we get,

$$\mu_{n,2}(x) \leq \frac{b_n}{n} x(B - 2A) + \frac{b_n^2}{n^2} C. \tag{2.2}$$

Lemma 2.5. For all $x \in [0,\infty)$, we have

$$\begin{aligned}\lambda_n(\frac{x}{b_n},\frac{t}{b_n}) &= \int_0^t K_n(\frac{x}{b_n},\frac{u}{b_n})du \\ &\le \frac{1}{(x-t)^2}\left(\frac{b_n}{n}x(B-2A)+\frac{b_n^2}{n^2}C\right),\end{aligned}$$

where

$$K_n(\frac{x}{b_n},\frac{u}{b_n}) = \sum_{k=0}^{\infty} P_{n,k}(\frac{x}{b_n})b_{n,k}(\frac{u}{b_n}).$$

Proof.

$$\begin{aligned}\lambda_n(\frac{x}{b_n},\frac{t}{b_n}) &= \int_0^t K_n(\frac{x}{b_n},\frac{u}{b_n})du \\ &\le \int_0^t K_n(\frac{x}{b_n},\frac{u}{b_n})\left(\frac{x-u}{x-t}\right)^2 du \\ &= \frac{1}{(x-t)^2}\int_0^t K_n(\frac{x}{b_n},\frac{u}{b_n})(x-u)^2du. \\ &= \frac{1}{(x-t)^2}\mu_{n,2}(x).\end{aligned}$$

By (3.1), we have,

$$\begin{aligned}\lambda_n(\frac{x}{b_n},\frac{t}{b_n}) &\le \frac{1}{(x-t)^2}\left(\frac{b_n^2}{n(n-1)}\{\frac{n^2x^2}{b_n^2}+\frac{nx}{b_n}B+C\}-\frac{2b_nx}{n}A-x^2\right) \\ &\le \frac{1}{(x-t)^2}\left(\frac{b_n}{n}x(B-2A)+\frac{b_n^2}{n^2}C\right).\end{aligned}$$

Q.E.D.

Set

$$Q^{\alpha}_{n,k}(\frac{x}{b_n}) = J^{\alpha}_{n,k}(\frac{x}{b_n}) - J^{\alpha}_{n,k+1}(\frac{x}{b_n}),$$

$$J^{\alpha}_{n,k}(\frac{x}{b_n}) = \left(\sum_{k=j}^{\infty} P_{n,j}(\frac{x}{b_n})\right)^{\alpha},$$

where $J^{\alpha}_{n,n+1}(\frac{x}{b_n}) = 0$ and $\alpha \ge 1$.

Lemma 2.6. [17]. Let j be a fixed non negative integer and $H(j) = \frac{(j+1/2)^{j+1/2}}{j!}e^{-(j+1/2)}$. Then, for all k, x such that $k \ge j$ and $x \in [0,\infty)$, there holds

$$Q_{n,k}(\frac{x}{b_n}) \le P_{n,k}(\frac{x}{b_n}) \le \frac{H(j)}{\sqrt{\frac{nx}{b_n}}}.$$

Moreover, the cofficient $H(j)$ and the asymptotic order $n^{-1/2}$ for $(n \to \infty)$ are the best possible.

3 Main result

In this section, by means of the techniques of probability theory, we shall estimate the rate of convergence of operators $L_n^*(f;x)$, for functions of bounded variation in terms of the Chanturiya's modulus of variation at the points on which one sided limit $f(x\pm)$ exist. For the sake of brevity, let the auxiliary function g_x be defined by

$$g_x(t) = \begin{cases} f(t) - f(x+), & x < t < \infty, \\ 0, & t = x, \\ f(t) - f(x-), & 0 \le t < x. \end{cases}$$

Lemma 3.1. [4]. For all $x \in [0, \infty)$, we have

$$\int_x^\infty b_{n,k}(\frac{t}{b_n}) = \sum_{j=0}^{k} P_{n,j}(\frac{x}{b_n}).$$

The main theorem of this paper is as follows.

Theorem 3.2. Let f be a function of bounded variation on every finite subinterval of $[0, \infty)$. Then for every $x \in (0, \infty)$, and n sufficiently large, we have,

$$\left|L_n^*(f;x) - \frac{1}{2}(f(x+) + f(x-))\right| \le \left(\frac{4F_n(x)}{x^2} + \frac{1}{n}\right)\sum_{k=1}^{n}\bigvee_{x-\frac{x}{\sqrt{k}}}^{x}(g_x) + \frac{2H(j)}{\sqrt{\frac{nx}{b_n}}} \mid f(x+) - f(x-) \mid$$
$$+\frac{(2x)^\gamma}{x^{2m}} O(n^{-(m+1)/2}),$$

where $F_n(x) = \frac{b_n}{n}x(B-2A) + \frac{b_n^2}{n^2}C$ and $\bigvee_a^b(g_x)$ is the total variation of g_x on $[a, b]$.

Proof. For any $f \in BV[0, \infty)$, it can be seen easily that

$$f(t) = \frac{1}{2}(f(x+)+f(x-))+g_x(t)+\frac{f(x+) - f(x-)}{2}\,\mathrm{sgn}(t-x)+\delta_x(t)\left[f(x)-\frac{1}{2}(f(x+)+f(x-))\right],$$

where

$$\delta_x(t) = \begin{cases} 1, & x = t, \\ 0, & x \ne t. \end{cases} \tag{4.2}$$

If we apply the operator L_n^* to both sides of the equality (4.1), we have

$$L_n^*(f;x) = \frac{1}{2}(f(x+) + f(x-))L_n^*(1;x) + L_n^*(g_x;x) + \frac{f(x+) - f(x-)}{2} L_n^*(\mathrm{sgn}(t-x);x)$$
$$+\left[f(x) - \frac{1}{2}(f(x+) + f(x-))\right] L_n^*(\delta_x;x).$$

Hence, by Lemma(2.3) $L_n^*(1;x) = 1$, we get,

$$\left|L_n^*(f;x) - \frac{1}{2}(f(x+) + f(x-))\right| \le \mid L_n^*(g_x;x) \mid + \left|\frac{f(x+) - f(x-)}{2}\right| \mid L_n^*(\mathrm{sgn}(t-x);x) \mid$$
$$+\left|f(x) - \frac{1}{2}(f(x+) + f(x-))\right| \mid L_n^*(\delta_x;x) \mid .$$

For operators L_n^*, using (4.2) we can see that $L_n^*(\delta_x; x) = 0$. Hence, we have

$$\left| L_n^*(f;x) - \frac{1}{2}(f(x+) + f(x-)) \right| \leq \mid L_n^*(g_x;x) \mid + \left| \frac{f(x+) - f(x-)}{2} \right| \mid L_n^*(\mathrm{sgn}(t-x);x) \mid .$$

In order to prove above inequality, we need the estimates for $L_n^*(g_x;x)$ and $L_n^*(\mathrm{sgn}(t-x);x)$. We first estimate $\mid L_n^*(g_x;x) \mid$ as follows:

$$\begin{aligned}
\mid L_n^*(g_x;x) \mid &= \left| \sum_{k=0}^{\infty} P_{n,k}(\frac{x}{b_n}) \int_0^{\infty} b_{n,k}(\frac{t}{b_n}) g_x(t)dt \right| \\
&= \left| \sum_{k=0}^{\infty} P_{n,k}(\frac{x}{b_n}) \left(\int_0^{x-\frac{x}{\sqrt{n}}} + \int_{x-\frac{x}{\sqrt{n}}}^{x+\frac{b_n-x}{\sqrt{n}}} + \int_{x+\frac{b_n-x}{\sqrt{n}}}^{2x} + \int_{2x}^{\infty} \right) b_{n,k}(\frac{t}{b_n}) g_x(t)dt \right| \\
&\leq \left| \sum_{k=0}^{\infty} P_{n,k}(\frac{x}{b_n}) \int_0^{x-\frac{x}{\sqrt{n}}} b_{n,k}(\frac{t}{b_n}) g_x(t)dt \right| \\
&\quad + \left| \sum_{k=0}^{\infty} P_{n,k}(\frac{x}{b_n}) \int_{x-\frac{x}{\sqrt{n}}}^{x+\frac{b_n-x}{\sqrt{n}}} b_{n,k}(\frac{t}{b_n}) g_x(t)dt \right| \\
&\quad + \left| \sum_{k=0}^{\infty} P_{n,k}(\frac{x}{b_n}) \int_{x+\frac{b_n-x}{\sqrt{n}}}^{2x} b_{n,k}(\frac{t}{b_n}) g_x(t)dt \right| \\
&\quad + \left| \sum_{k=0}^{\infty} P_{n,k}(\frac{x}{b_n}) \int_{2x}^{\infty} \right) b_{n,k}(\frac{t}{b_n}) g_x(t)dt \right| \\
&= \mid I_1(n,x) \mid + \mid I_2(n,x) \mid + \mid I_3(n,x) \mid + \mid I_4(n,x) \mid .
\end{aligned}$$

We shall evaluate $\mid I_1(n,x) \mid$, $\mid I_2(n,x) \mid$, $\mid I_3(n,x) \mid$ and $\mid I_4(n,x) \mid$. To do this, we first observe that $\mid I_1(n,x) \mid$, $\mid I_2(n,x) \mid$, $\mid I_3(n,x) \mid$ and $\mid I_4(n,x) \mid$ can be written as Lebesgue-Stieltjes integral,

$$\begin{aligned}
I_1(n,x) &= \left| \int_0^{x-\frac{x}{\sqrt{n}}} g_x(t) d_t \lambda_n(\frac{x}{b_n}, \frac{t}{b_n}) \right| \\
I_2(n,x) &= \left| \int_{x-\frac{x}{\sqrt{n}}}^{x+\frac{b_n-x}{\sqrt{n}}} g_x(t) d_t \lambda_n(\frac{x}{b_n}, \frac{t}{b_n}) \right| \\
I_3(n,x) &= \left| \int_{x+\frac{b_n-x}{\sqrt{n}}}^{2x} g_x(t) d_t \lambda_n(\frac{x}{b_n}, \frac{t}{b_n}) \right| \\
I_4(n,x) &= \left| \int_{2x}^{\infty} g_x(t) d_t \lambda_n(\frac{x}{b_n}, \frac{t}{b_n}) \right|,
\end{aligned}$$

where

$$\lambda_n(\frac{x}{b_n}, \frac{t}{b_n}) = \int_0^t K_n(\frac{x}{b_n}, \frac{u}{b_n}) du$$

and

$$K_n(\frac{x}{b_n}, \frac{t}{b_n}) = \sum_{k=0}^{\infty} P_{n,k}(\frac{x}{b_n}) b_{n,k}(\frac{t}{b_n}).$$

First, we estimate $I_2(n,x)$. For $t \in [x - \frac{x}{\sqrt{n}}, x + \frac{b_n - x}{\sqrt{n}}]$, we have

$$\begin{aligned} | I_2(n,x) | &\leq \left| \int_{x-\frac{x}{\sqrt{n}}}^{x+\frac{b_n-x}{\sqrt{n}}} (g_x(t) - g_x(x)) d_t \lambda_n(\frac{x}{b_n}, \frac{t}{b_n}) \right| \\ &\leq \int_{x-\frac{x}{\sqrt{n}}}^{x+\frac{b_n-x}{\sqrt{n}}} | (g_x(t) - g_x(x)) d_t | \left| \lambda_n(\frac{x}{b_n}, \frac{t}{b_n}) \right| \\ &\leq \bigvee_{x-\frac{x}{\sqrt{n}}}^{x} (g_x) \leq \frac{1}{n} \sum_{k=1}^{n} \bigvee_{x-\frac{x}{\sqrt{k}}}^{x} (g_x). \end{aligned}$$

Next, we estimate $I_1(n,x)$. Using partial Lebesgue-Stieltjes integration, we obtain

$$\begin{aligned} I_1(n,x) &= \int_0^{x-\frac{x}{\sqrt{n}}} g_x(t) d_t \lambda_n(\frac{x}{b_n}, \frac{t}{b_n}) \\ &= g_x(x - \frac{x}{\sqrt{n}}) \lambda_n(\frac{x}{b_n}, \frac{x - \frac{x}{\sqrt{n}}}{b_n}) - g_x(0) \lambda_n(\frac{x}{b_n}, 0) \\ &\quad - \int_0^{x-\frac{x}{\sqrt{n}}} \lambda_n(\frac{x}{b_n}, \frac{t}{b_n}) d_t(g_x(t)). \end{aligned}$$

Since

$$\left| g_x(x - \frac{x}{\sqrt{n}}) \right| = \left| g_x(x - \frac{x}{\sqrt{n}}) - g_x(x) \right| \leq \bigvee_{x-\frac{x}{\sqrt{n}}}^{x} (g_x),$$

it follows that

$$| I_1(n,x) | \leq \bigvee_{x-\frac{x}{\sqrt{n}}}^{x} (g_x) \left| \lambda_n(\frac{x}{b_n}, \frac{x - \frac{x}{\sqrt{n}}}{b_n}) \right| + \int_0^{x-\frac{x}{\sqrt{n}}} \lambda_n(\frac{x}{b_n}, \frac{t}{b_n}) d_t(-\bigvee_{t}^{x}(g_x)).$$

From Lemma (2.5), it is clear that

$$\lambda_n(\frac{x}{b_n}, \frac{x - \frac{x}{\sqrt{n}}}{b_n}) \leq \frac{1}{(\frac{x}{\sqrt{n}})^2} \left\{ \frac{b_n}{n} x(B - 2A) + \frac{b_n^2}{n^2} C \right\}.$$

It follows that

$$\begin{aligned} | I_1(n,x) | &\leq \bigvee_{x-\frac{x}{\sqrt{n}}}^{x} (g_x) \frac{1}{(\frac{x}{\sqrt{n}})^2} \left\{ \frac{b_n}{n} x(B - 2A) + \frac{b_n^2}{n^2} C \right\} \\ &+ \int_0^{x-\frac{x}{\sqrt{n}}} \frac{1}{(x-t)^2} \left\{ \frac{b_n}{n} x(B - 2A) + \frac{b_n^2}{n^2} C \right\} d_t(-\bigvee_{t}^{x}(g_x)) \\ &= \bigvee_{x-\frac{x}{\sqrt{n}}}^{x} (g_x) \frac{F_n(x)}{(\frac{x}{\sqrt{n}})^2} + F_n(x) \int_0^{x-\frac{x}{\sqrt{n}}} \frac{1}{(x-t)^2} d_t(-\bigvee_{t}^{x}(g_x)). \end{aligned}$$

Furthermore, since

$$\int_0^{x-\frac{x}{\sqrt{n}}} \frac{1}{(x-t)^2} d_t(-\bigvee_t^x(g_x)) = -\frac{1}{(x-t)^2}\bigvee_t^x(g_x)\Big|_0^{x-\frac{x}{\sqrt{n}}} + \int_0^{x-\frac{x}{\sqrt{n}}} \frac{2}{(x-t)^3}\bigvee_t^x(g_x)dt$$
$$= -\frac{1}{(\frac{x}{\sqrt{n}})^2}\bigvee_{x-\frac{x}{\sqrt{n}}}^x(g_x) + \frac{1}{x^2}\bigvee_0^x(g_x) + \int_0^{x-\frac{x}{\sqrt{n}}} \frac{2}{(x-t)^3}\bigvee_t^x(g_x)dt.$$

Putting $t = x - \frac{x}{\sqrt{u}}$ in the last integral, we get

$$\int_0^{x-\frac{x}{\sqrt{n}}} \frac{2}{(x-t)^3}\bigvee_t^x(g_x)dt = \frac{1}{x^2}\int_1^n \bigvee_{x-\frac{x}{\sqrt{u}}}^x(g_x)du = \frac{1}{x^2}\sum_{k=1}^n \bigvee_{x-\frac{x}{\sqrt{k}}}^x(g_x).$$

Consequently,

$$|\, I_1(n,x)\,| \le \frac{F_n(x)}{(\frac{x}{\sqrt{n}})^2}\bigvee_{x-\frac{x}{\sqrt{n}}}^x(g_x) + F_n(x)\Big\{-\frac{1}{(\frac{x}{\sqrt{n}})^2}\bigvee_{x-\frac{x}{\sqrt{n}}}^x(g_x) + \frac{1}{x^2}\bigvee_0^x(g_x) + \frac{1}{x^2}\sum_{k=1}^n\bigvee_{x-\frac{x}{\sqrt{k}}}^x(g_x)\Big\}$$
$$= F_n(x)\Big\{\frac{1}{x^2}\bigvee_0^x(g_x) + \frac{1}{x^2}\sum_{k=1}^n\bigvee_{x-\frac{x}{\sqrt{k}}}^x(g_x)\Big\}$$
$$= \frac{2F_n(x)}{x^2}\sum_{k=1}^n\bigvee_{x-\frac{x}{\sqrt{k}}}^x(g_x).$$

Using the similar method for estimating $|\, I_3(n,x)\,|$, we get

$$|\, I_3(n,x)\,| \le F_n(x)\Big\{\frac{1}{x^2}\bigvee_0^x(g_x) + \frac{1}{x^2}\sum_{k=1}^n\bigvee_{x-\frac{x}{\sqrt{k}}}^x(g_x)\Big\}$$
$$\le \frac{2F_n(x)}{x^2}\sum_{k=1}^n\bigvee_{x-\frac{x}{\sqrt{k}}}^x(g_x).$$

By assumption $g_x(t) = O(t^\gamma)$ for $\gamma > 0$ at $t \to \infty$, taking $m \ge \gamma/2$ and $t \ge 2x$, $\frac{t^\gamma}{(t-x)^{2m}}$ is monotone decreasing for variable t, hence by Lemma (2.5), we get

$$|\, I_4(n,x)\,| \le \Big|\int_{2x}^\infty g_x(t)d_t\lambda_n(\frac{x}{b_n},\frac{t}{b_n})\Big|$$
$$= \sum_{k=0}^\infty P_{n,k}(\frac{x}{b_n})\int_{2x}^\infty b_{n,k}(\frac{t}{b_n})t^\gamma dt$$
$$= \frac{(2x)^\gamma}{x^{2m}}\sum_{k=0}^\infty P_{n,k}(\frac{x}{b_n})\int_{2x}^\infty (t-x)^{2m}b_{n,k}(\frac{t}{b_n})dt$$
$$= \frac{(2x)^\gamma}{x^{2m}}\mu_{n,m}(x)$$
$$= \frac{(2x)^\gamma}{x^{2m}}O(n^{-(m+1)/2}).$$

Hence, from (4.3)-(4.6), it follows that

$$\begin{aligned}
| L_n^*(g_x;x) | &\leq | I_1(n,x) | + | I_2(n,x) | + | I_3(n,x) | + | I_4(n,x) | \\
&\leq \frac{2F_n(x)}{x^2}\sum_{k=1}^{n}\bigvee_{x-\frac{x}{\sqrt{k}}}^{x}(g_x) + \frac{1}{n}\sum_{k=1}^{n}\bigvee_{x-\frac{x}{\sqrt{k}}}^{x}(g_x) + \frac{2F_n(x)}{x^2}\sum_{k=1}^{n}\bigvee_{x-\frac{x}{\sqrt{k}}}^{x}(g_x) \\
&+ \frac{(2x)^\gamma}{x^{2m}}O(n^{-(m+1)\backslash 2}) \\
&\leq \left\{\frac{4F_n(x)}{x^2} + \frac{1}{n}\right\}\sum_{k=1}^{n}\bigvee_{x-\frac{x}{\sqrt{k}}}^{x}(g_x) + \frac{(2x)^\gamma}{x^{2m}}O(n^{-(m+1)\backslash 2}).
\end{aligned}$$

Now secondly, we can estimate $L_n^*(\text{sgn}(t-x);x)$. If we apply operators L_n^* to the signum function and using Lemma (3.1), we have

$$\begin{aligned}
L_n^*(\text{sgn}(t-x);x) &= -1 + 2\sum_{k=0}^{\infty}P_{n,k}(\frac{x}{b_n})\int_x^\infty b_{n,k}(\frac{t}{b_n})dt \\
&= -1 + 2\sum_{k=0}^{\infty}P_{n,k}(\frac{x}{b_n})\sum_{k=j}^{k}p_{n,j}(\frac{x}{b_n}) \\
&= -1 + 2\sum_{j=0}^{\infty}p_{n,j}(\frac{x}{b_n})\sum_{k=j}^{\infty}P_{n,k}(\frac{x}{b_n}) \\
&= -1 + 2\sum_{j=0}^{\infty}p_{n,j}(\frac{x}{b_n})J_{n,j}(\frac{x}{b_n}).
\end{aligned}$$

Thus,

$$L_n^*(\text{sgn}(t-x);x) = 2\sum_{j=0}^{\infty}p_{n,j}(\frac{x}{b_n})J_{n,j}(\frac{x}{b_n}) - \sum_{j=0}^{\infty}Q_{n,j}^2(\frac{x}{b_n})$$

since $\sum_{k=0}^{\infty}Q_{n,k}(\frac{x}{b_n}) = 1$. By mean value theorem, we have

$$Q_{n,j}^2(\frac{x}{b_n}) = J_{n,j}^2(\frac{x}{b_n}) - J_{n,j+1}^2(\frac{x}{b_n}) = 2p_{n,j}(\frac{x}{b_n})\gamma_{n,j}(\frac{x}{b_n}),$$

where

$$J_{n,j}(\frac{x}{b_n}) < \gamma_{n,j}(\frac{x}{b_n}) < J_{n,j}(\frac{x}{b_n}).$$

Therefore,

$$\begin{aligned}\left|L_n^*(\operatorname{sgn}(t-x);x)\right| &= 2\sum_{j=0}^{\infty} p_{n,j}(\frac{x}{b_n})(J_{n,j}(\frac{x}{b_n})-\gamma_{n,j}(\frac{x}{b_n}))\\ &= 2\sum_{j=0}^{\infty} p_{n,j}(\frac{x}{b_n})(J_{n,j}(\frac{x}{b_n})-J_{n,j+1}(\frac{x}{b_n}))\\ &= 2\sum_{j=0}^{\infty} p_{n,j}^2(\frac{x}{b_n}).\end{aligned}$$

Using Lemma (2.6), we have

$$\left|L_n^*(\operatorname{sgn}(t-x);x)\right| \leq \frac{2H(j)}{\sqrt{\frac{nx}{b_n}}}\sum_{j=0}^{\infty} p_{n,j}(\frac{x}{b_n}) = \frac{2H(j)}{\sqrt{\frac{nx}{b_n}}}. \tag{4.8}$$

Combining (4.7) and (4.8), we get the required result.

This completes the proof of the theorem. Q.E.D.

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References

[1] R. Bojanic, M. Vuilleunier, *On the rate of convergence of Fourier Legendre series of functions of bounded variation*, J. Approx. Theory, 31 (1981) 67-79.

[2] İ. Büyükyazıcı, H. Tanberkan, S. Kirci Serenbay, Ç. Atakut, *Approximation by Chlodowsky type Jakimovski-Leviatan operators*, J. Comp. App. Math., 259 (2014) 153-163.

[3] F. Cheng, *On the rate of convergence of Bernstein polynomials of functions of bounded variation*, J. Approx. Theory, 39 (1983) 259-274.

[4] V. Gupta, *Rate of convergence on Baskakov-Beta-Bézier operators for bounded variation functions*, Int. J. Math. Math. Sci., 32(8) (2002) 471-479.

[5] A. Jakimovski, D. Leviatan, *Generalized Szász operators for the approximation in the infinite interval*, Mathematica (Cluj), 11(34) (1969) 97-103.

[6] A. Karaisa, *Approximation by Durrmeyer type Jakimovski-Leviatan operators*, Math. Meth. Appl. Sci., 39 (2016): 2401-2410. doi: 10.1002/mma.3650.

[7] M. Mursaleen, Khursheed J. Ansari, *On Chlodowsky variant of Szász operators by Brenke type polynomials*, Appl. Math. Comput., 271 (2015) 991-1003.

[8] M. Mursaleen, K.J. Ansari, Asif Khan, *Approximation by a Kantorovich type q-Bernstein-Stancu operators*, Complex Analysis and Operator Theory, 11(1) (2017) 85–107.

[9] M. Mursaleen, Md. Nasiruzzaman, A. Alotaibi, *On modified Dunkl generalization of Szsz operators via q-calculus*, Jou. Ineq. Appl., (2017) 2017: 38.

[10] M. Mursaleen, S. Rahman and A. Alotaibi, *Dunkl generalization of q-Szász-Mirakjan-Kantorovich operators which preserve some test functions*, Jou. Ineq. Appl., (2016) 2016: 317.

[11] H. M. Srivastava, Z. Finta, V. Gupta, *Direct results for a certain family of summation-integral type operators*, Appl. Math. Comput., 190 (2007) 449-457.

[12] H. M. Srivastava, V. Gupta, *A certain family of summation-integral type operators*, Math. Comput. Model., 37 (2003) 1307-1315.

[13] H. M. Srivastava, V. Gupta, *Rate of convergence for the Bézier variant of the Bleimann-Butzer-Hahn operators*, Appl. Math. Lett., 18 (2005) 849-857.

[14] H. M. Srivastava, M. Mursaleen, Abdullah M. Alotaibi, Md. Nasiruzzaman, A. A. H. Al-Abied, *Some approximation results involving the q-Szász-Mirakjan-Kantorovich type operators via Dunkl's generalization*, Math. Meth. Appl. Sci., DOI:10.1002/mma.4397.

[15] H. M. Srivastava, X.-M. Zeng, *Approximation by means of the Szász-Bézier integral operators*, Internat. J. Pure Appl. Math., 14 (2004) 283-294.

[16] X.M. Zeng, W. Chen, *On the rate of convergence of the generalized Durrmeyer type operators for functions of bounded variation*, J. Approx. Theory, 102 (2000) 1-12.

[17] X.M. Zeng, J. N. Zhao, *Exact bounds for some basis function of approximation operators*, J. Inequal. Appl., 6(2001) 563-575.